ON A GENERALIZED FORMULATION FOR THE OPTIMIZATION OF CHORD AND TWIST ANGLE OF WIND BLADE

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1. Introduction

The classical Stewart's (1976) model is widely used in the wind blades design because that model presents low computational cost and easy implementation, especially for the case of small wind rotors (Vaz et al, 2009). However, in the structure of Stewart's (1976) model the axial induction factor in the wake is twice the inducing factor in the rotor plane, ignoring the generally form established in the work of Wilson and Lissaman (1974) for the influence of vortex wake when the machine operates at low tip-speed-ratios. For values of tip-speed-ratio smaller than 2, a wind rotor tends to operate at low speed even for significant wind speeds. This regime of operation occurs in multiple blade turbines, where the rotation is generally slow, but with high torques, and is widely used for water pumping (Vaz et al. 2011; Mesquita and Alves, 2000). Thus, this work proposes a new mathematical model of aerodynamic optimization based on the relationship between the axial induction factors in the rotor and in the wake, at lower tip-speed ratio. This general relationship was presented by Wilson and Lissaman (1974), where the vortex wake is considered in the calculation of the local power coefficient of the wind blade. Thus, it's possible to promote an improvement in aerodynamics of the wind turbine with the maximization of the local power coefficient.

2. The Mathematical Model

Wilson and Lissaman (1974) present a mathematical model that considers the vortex wake caused by the wind turbine for the calculation of its power coefficient. In that model, the induced speeds u and u_1 in the rotor plane and in the wake, respectively, are written as:

$$\begin{cases} V_0 - v = u \equiv (1-a)V_0 \\ V_0 - v_1 = u_1 \equiv (1-b)V_0 \end{cases} \quad \text{(eq. 1)}$$

where $v = aV_0$ and $v_1 = bV_0$. V_0 is the undisturbed flow speed, a and b are the axial induction factors in the rotor plane and in the wake, respectively.

Figure 1 illustrates the behavior of the flow in a stream tube (Hansen, 2008) and the flow axial speeds.

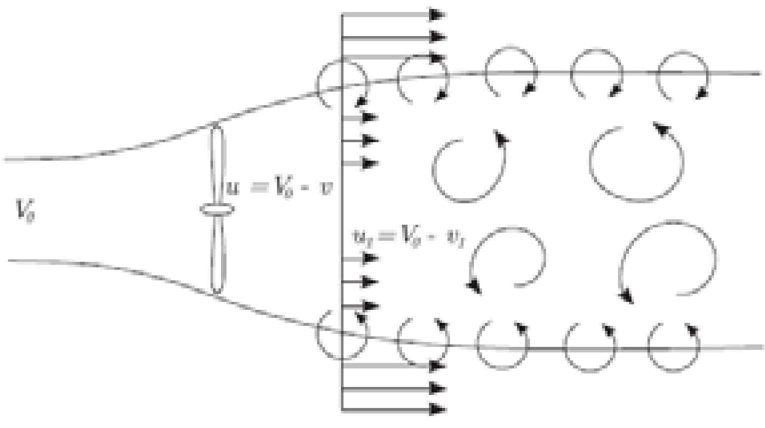

Fig. 1: Simplified illustration of the speeds in the rotor plane and in the wake (Hansen, 2008).

From Eq. (1) Wilson and Lissaman (1974) showed that it is possible to establish a general relationship between the axial induction factors *a* and *b* from the application of the continuity, momentum and energy equations for the induced speeds, where the effect of vortex wake formed downstream of the turbine is considered. In this model, the hypothesis that the wake behaves like a free vortex is considered. The result of this application is shown in Eq. (2), where the axial induction factor in the rotor plane has a non-linear relationship with the axial inducing factor in the wake for low tip-speed-ratios (*X*), especially for values $X < 2$, as shown in Fig. 2.

$$a = \frac{b}{2}\left[1 - \frac{b^2(1-a)}{4X^2(b-a)}\right] \qquad \text{(eq. 2)}$$

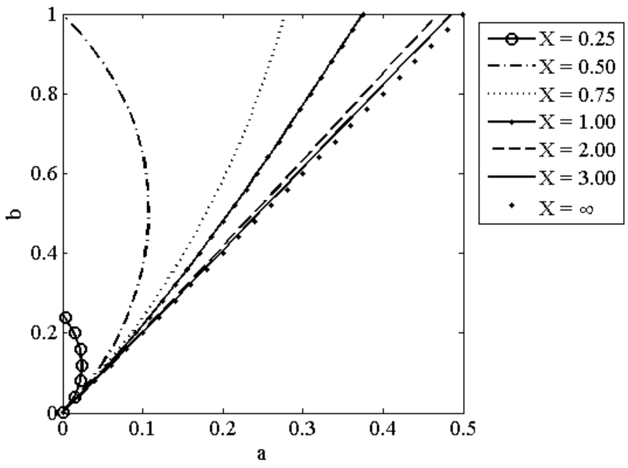

Fig. 2: Relation *b/a* for some values of *X*.

The local power coefficient in this case has the form:

$$Cp = \frac{b^2(1-a)^2}{b-a} \qquad \text{(eq. 3)}$$

Wilson and Lissaman (1974) showed that it is necessary to introduce a correction in Eq. (3), because the hypothesis that the wake formed behaves like a free vortex causes infinite speeds on the wake near the axis. This fact can be seen in Fig. 3, where the local power coefficient tends to high values when $X \leq 0.8$ and $a < 0.4$. *Cp* can take values greater than 1 when *X* is very small. For values of $0.81 < X < 2$, *Cp* can take maximum values less than 100% where $0 < a < 0.815$ and $0 < b < 2$. So, there is a valid region for $0.8 < Cp < 1$, where it is possible to use Eq. (3) to develop a new formulation for the aerodynamic optimization of wind rotors, especially for $a < 0.4$, where there is no need for corrections of the axial induction factor in the rotor plane, as described by Glauert (1935).

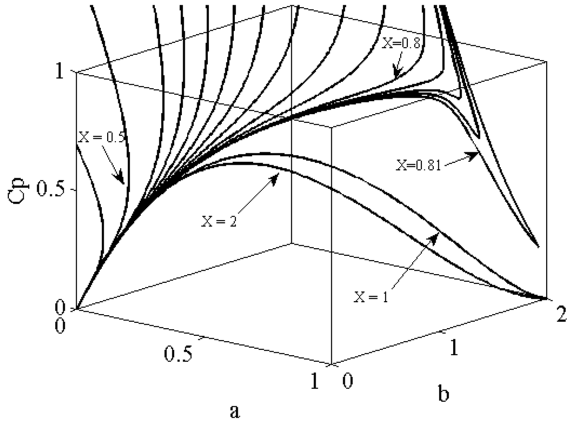

Fig. 3: Power coefficient as a function of axial induction factors *a* and *b*.

In an attempt to avoid the problem of infinite speeds near the axis, Wilson and Lissaman (1974) rewrote Eq. (3) according to the Rankine vortex $N \equiv \Omega / w_{\max}$.

$$Cp = \frac{b\left(1-a\right)^{2}}{b-a}\left[2Na + \left(1-N\right)b\right] \tag{eq. 4}$$

where $w_{\max}$ is the maximum speed of the vortex wake and W is the rotor angular velocity. However, there is a physical limitation for determining the angular velocity in the vortex wake. In the case of a wind rotor, it is not easy to determine $w_{\max}$. Figs. 4 and 5 show the behavior of the local power coefficient for N equal to 1 and 2, respectively. The value of the maximum local power coefficient of a rotor is heavily influenced by $w_{\max}$ in the region where $X < 2$. It is observed that for $X \geq 2$ there is no influence of vortex wake in the calculation of power coefficient of the turbine. However, to use this model, there is a need for accurately determining the maximum angular velocity in the rotor wake.

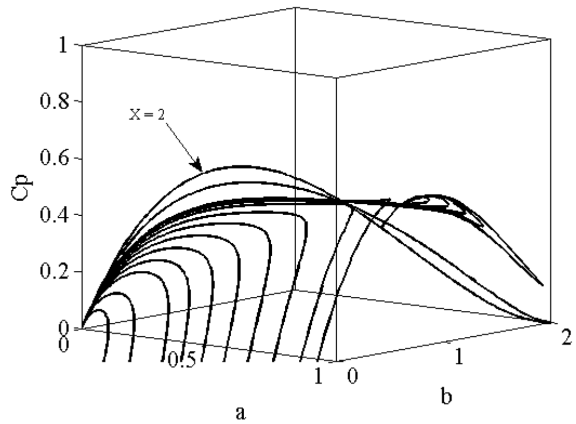

Fig. 4: Power coefficient as a function of axial induction factors *a* and *b* for *N* = 1.

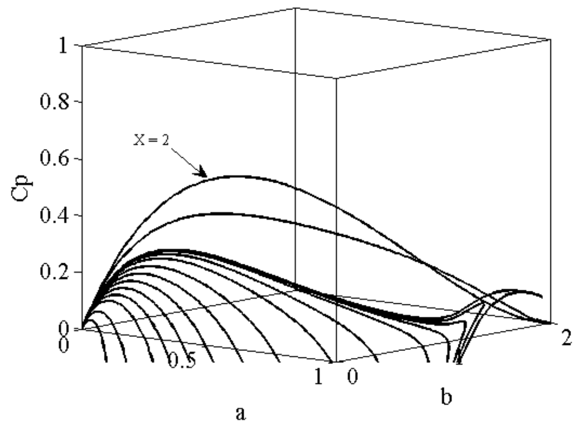

Fig. 5: Power coefficient as a function of axial induction factors *a* and *b* for *N* = 2.

Thus, the present work uses Eq. (3) to determine a new equation that maximizes the local power coefficient, *Cp*, using a criterion for the initial value of the axial induction factor *a*, to avoid the effect of infinite speeds near the axis. Therefore, it is possible to use Eq. (3), assigning $dCp/da = 0$, to obtain an explicit relationship that determines the value of the optimum axial induction factor in the rotor plane. So, in this case, there are three analytical roots, in which only one has physically consistent results:

$$a_{opt} = \frac{b\left(-1+\frac{db}{da}\right)+2\frac{db}{da}-\sqrt{4\left(\frac{db}{da}\right)^2-4b\frac{db}{da}\left(3+\frac{db}{da}\right)+b^2\left[1+14\frac{db}{da}+\left(\frac{db}{da}\right)^2\right]}}{4\frac{db}{da}} \qquad \text{(eq. 5)}$$

where db/da is obtained by differentiating Eq. (2).

$$\frac{db}{da} = \frac{8X^2\left(a-b\right)^2-b^3\left(b-1\right)}{4X^2\left(a-b\right)^2-ab^2\left(a-1\right)} \qquad \text{(eq. 6)}$$

Using Eq. (5) the optimal values of the axial induction factor that locally maximizes the power coefficient is computed. Equation (5) reduces to the optimal value of the actuator disk theory, assigning $b = 2a$ and hence $db/da = 2$ for $X > 2$, where $a_{opt} = 1/3$ (Eggleston and Stoddard, 1987). Figure 6 shows the curves for the induction factors for the local tip-speed-ratio, where *r* is the radial position. It is observed that the behavior of the parameters *b* and *b*' are different from those used by Glauert (1935), $b = 2a$ and $b' = 2a'$, where the induction factors in the wake are twice the induction factors in the rotor plane for any value of $\Omega r/V_0$. Glauert's (1935) model does not consider the nonlinearity given by Eq. (2) when $\Omega r/V_0 < 2$, as seen in Fig. 6. Figure 7 shows that the optimization proposed results in a tendency of the local power coefficient reaching maximum efficiency for an ideal turbine, according to Betz (1926) limit, in which the maximum energy extracted from the wind is about 59.26 %.

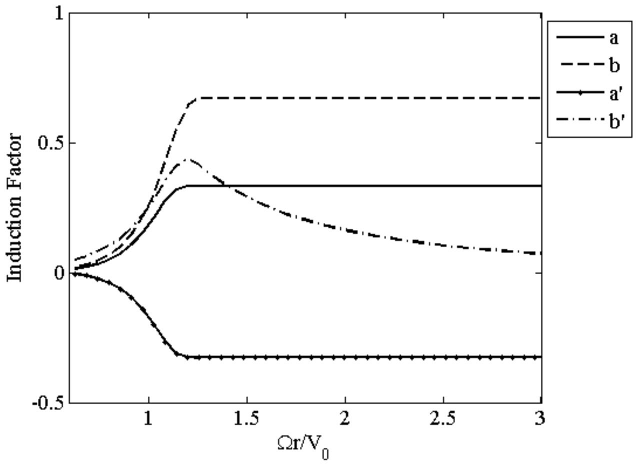

Fig. 6: Behavior of the axial induction factor in the rotor plane and in the wake.

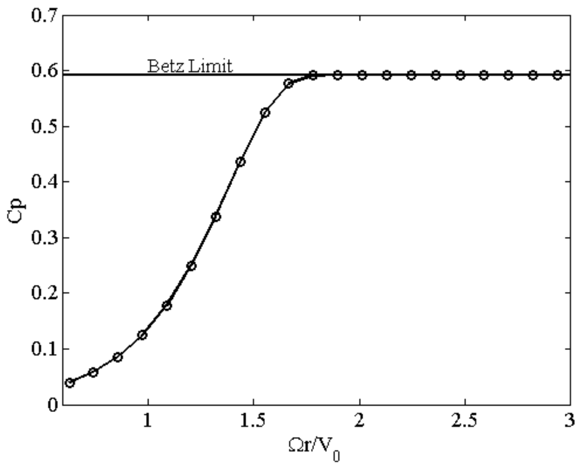

Fig. 7: Behavior of the local power coefficient as a function of local tip-speed-ratio.

Thus, with the value a_{opt} calculated in Eq. (5), the optimal chord is calculated with Eq. (7), which can be obtained from Vaz et al. (2011).

$$c_{opt} = \frac{4\pi r b F \sin^2(\phi)}{B C_n (1 - a_{opt})} \quad \text{(eq. 7)}$$

where F is the Prandtl (as described by Vaz et al., 2011) tip-loss factor, ϕ is the flow angle, and B is the number of blades.

$$C_n = C_l \cos\phi + C_d \sin\phi \quad \text{(eq. 8)}$$

C_l and C_d are the lift and drag coefficients, respectively, which are usually obtained from wind tunnel tests. For the calculation of a' the relationship given by Eq. (9) is used, obtained from the velocity diagram shown in Fig. 8. Equation (9) can be verified in the work of Hansen (2008).

$$a'_{opt} = \left[\frac{\left(1 - a_{opt}\right)}{x \tan \phi}\right] - 1 \qquad \text{(eq. 9)}$$

where the local tip-speed-ratio is:

$$x = \frac{\Omega r}{V_0} \qquad \text{(eq. 10)}$$

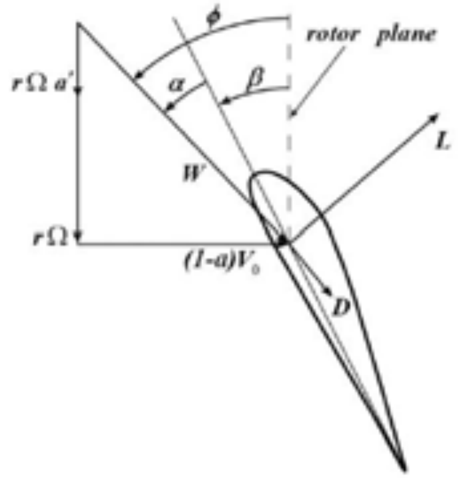

Fig. 8: Velocity diagram for a rotor blade section.

The use of Eq. (9) is valid even for values of $X < 2$, since this relation is obtained in the rotor plane and not in the wake. Once a'_{opt} is calculated, b' is estimated by Eq. (11), which can be verified in the work of Vaz et al. (2011).

$$b'_{opt} = \frac{\left(1 + a'_{opt}\right)\sigma C_t}{2F \sin\phi \cos\phi} \qquad \text{(eq. 11)}$$

where σ is the solidity of the turbine

$$\sigma = \frac{c_{opt} B}{2\pi r} \qquad \text{(eq. 12)}$$

and

$$C_t = C_l \sin\phi - C_d \cos\phi \qquad \text{(eq. 13)}$$

The optimum flow angle, ϕ, is calculated assigning

$$\phi_{opt} = \tan^{-1}\left[\frac{\left(1 - a_{opt}\right)}{\left(1 + a'_{opt}\right)x}\right] \qquad \text{(eq. 14)}$$

Finally, the optimum twist angle is given by:

$$\beta_{opt} = \phi_{opt} - \alpha \quad \text{(eq. 15)}$$

3. Results and Discussions

The results here obtained consider a small wind turbine with 4 m diameter and 0.4 m hub diameter, constant rotation of 120 rpm, average wind speed of 4 m / s and 4 blades. The airfoil used is the NACA 64_4-421 (Abbot and Doenhoff, 1959).

The proposed model is based on the hypothesis that the rotational speed of the wake near the axis can reach infinite values, which takes the local power coefficient also to infinity near the axis, with a consequent undefined increase of chord. However, this work proposes to control the axial induction factor a in the chord calculation near the axis. Figure 9 shows the behavior of the mathematical model for various values of a, where is possible to verify the effect of increasing the chord near the axis for a = 1/3. Note that when a tends to the value 1/3, the chord distribution and the twist angle distribution, Figs. 9 and 10, tend to the classical model of Stewart (1976), but only for radial positions $r/R > 0.3$. This fact occurs because the local tip-speed-ratio is very small near the axis, in this case $\Omega r / V_0 < 2$.

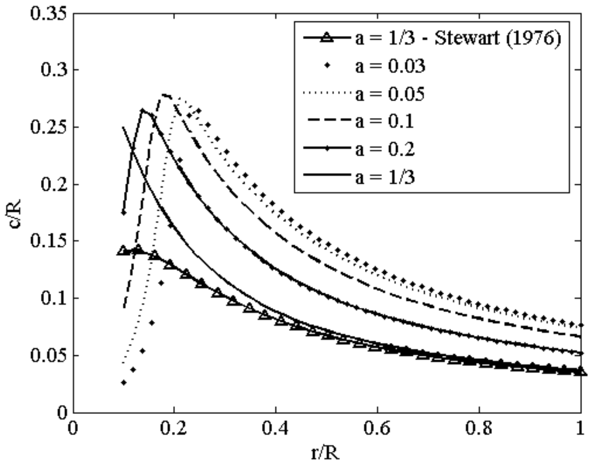

Fig. 9: Chord distribution for some values of axial induction factor in the rotor plane.

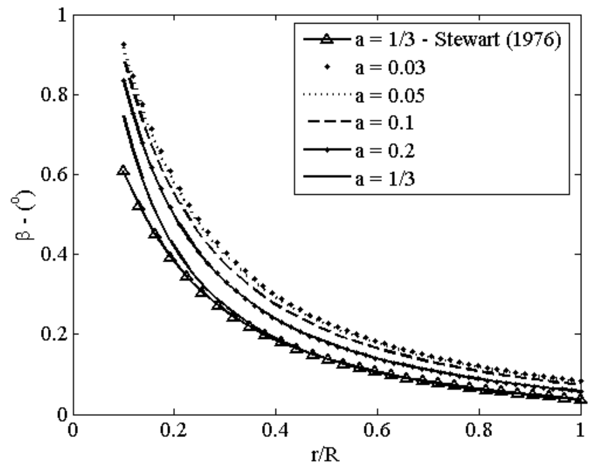

Fig. 10: Twist angle distribution for some values of axial induction factor in the rotor plane.

In the case of Stewart's (1976) model, it is not possible to take into account the same criterion, since the relationship between the induction factors in the rotor plan and in the wake do not depend on X.

Figure 11 and 12 show the optimized blades using the proposed model and Stewart´s (1976). The blades were obtained under the same design conditions. It is observed that the blade designed with the proposed model presents a larger chord than the one designed with Stewart's model (1976). This occurs because the tangential induction factor a' is very small, as can be seen in Fig. 6, which causes a considerable increase in the flow angle, and hence in the chord, that varies directly with the term $\sin^2(\phi)$ in Eq. (7). This effect occurs also with Eq. (15) in the case of twist angle. The blade shown in Fig. 11 is designed starting with the induction factor value of 0.1 for the first chord of the blade counting from its root. With this criterion it is possible to ensure that the local power coefficient is always below the Betz (1926) limit (see Fig. 3), for any value of X. Thus, the criterion of the inducing factor controls the optimal chord and twist angle and should be initiated with $a < 1/3$, since it is possible to ensure that the chord near the axis has an attenuated behavior, as shown in Fig. 9. Therefore, the effect of reducing the chord near the axis is a consequence of this criterion (see Fig. 11). For comparison, Fig. 12 shows the blade constructed using the Stewart's (1976) model. The comparison shows significant differences in the aerodynamic shapes of the blades constructed, showing that the proposed model presents physically consistent solutions, as presented in Figs. 11 and 12.

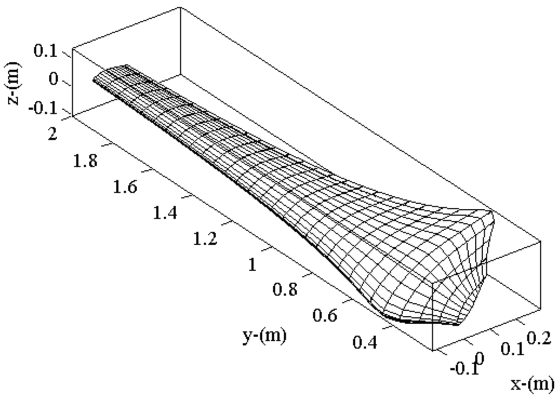

Fig. 11: Blade designed with the proposed model.

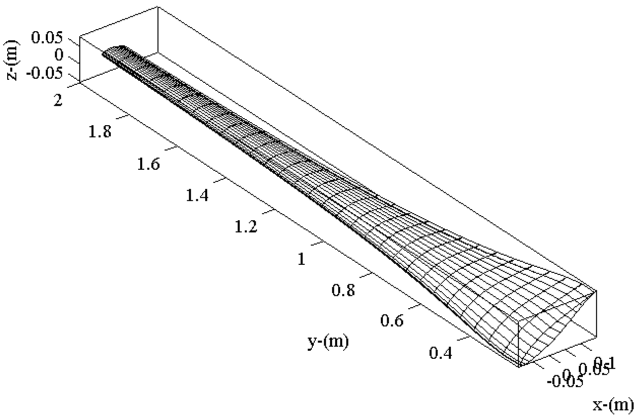

Fig. 12: Blade designed with Stewart's optimization model.

The calculation of turbine efficiency was performed using the classical Glauert' (1935) model for both turbines designed. Figs. 13, 14 and 15 show the results. Figure 13 shows the turbine´s total power coefficient, calculated by Eq. (16), which is given by integrating the local power coefficients (Hansen, 2008).

$$C_P^T = \frac{P}{\frac{1}{2}\rho V_0^3 A} = \frac{8}{X^2}\int_0^X a'F(1-aF)x^3dx \qquad \text{(eq. 16)}$$

where A is the rotor´s swept area and P is the power developed by the turbine.

$$P = 4\pi\Omega^2 V_0 \int_0^R a' F(1 - aF) r^3 dr \quad \text{(eq. 17)}$$

It is observed that the total power coefficient of the turbine constructed with the proposed model tends to the result obtained with the classical Stewart's (1976) model, when the axial induction factor approaches 1/3, which is the optimum value for an ideal turbine. The best efficiency curves in this case are obtained when the blade is designed using the proposed model associated with the criterion of axial induction factor in calculating the distributions of chord and twist angle of the wind blade. This fact leads to the modification of the aerodynamic shape of the wind blade. Figure 14 shows the effect of the proposed model for a low tip-speed-ratio region, where the results show that the power curves are improved also in this region. Thus, the model described in this work gives good results for multiple blades turbines, which are widely used in water pumping systems.

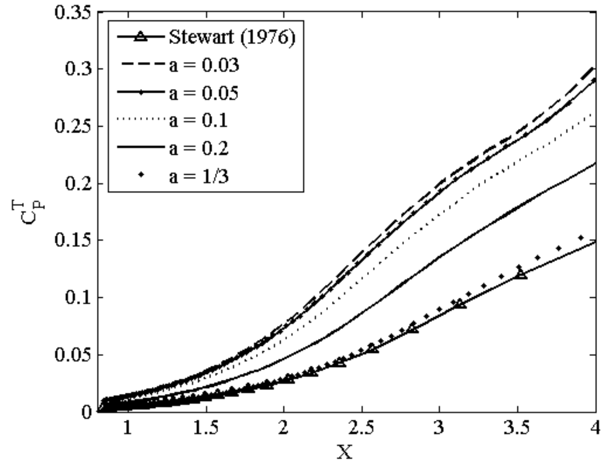

Fig. 13: Power coefficient as a function of tip-speed-ratio.

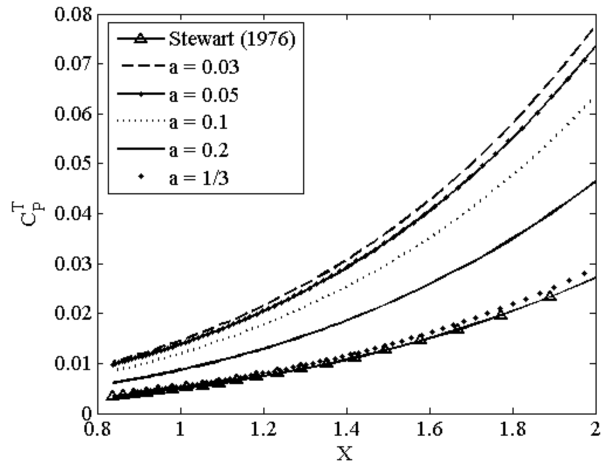

Fig. 14: Effects of the model at low tip-speed-ratios.

Figure 15 shows the difference between the output power of the designed wind turbines for multiple starting values of the axial induction factor, where the turbine developed with the proposed model provides better efficiency when compared with Stewart's (1976) model.

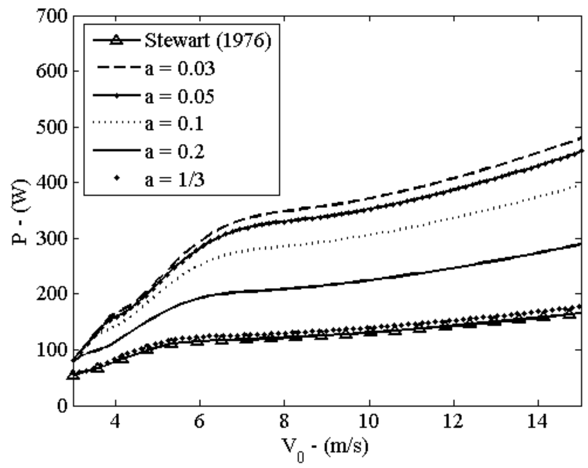

Fig. 15: Output power as a function of wind speed.

4. Conclusion

The proposed mathematical model represents an alternative tool for the optimal design of wind rotors, where the main advantage is the use of a general relationship between the induction factors in the rotor plane and in the vortex wake described by Wilson and Lissaman (1974), taking into account the effects of the wake rotation. The proposed method shows better efficiency when compared with the classical Stewart's (1976) optimization, satisfying the condition described by Betz (1926), where the maximum energy to be extracted from the flow is 59.26%. The behavior of the induction factors in the wake is completely non-linear for low tip-speed-ratios, stressing the need for formulations that meet these characteristics. This fact considers the slow operation of a turbine, such as multiple blades, commonly used in water pumping systems. The comparisons show that the developed model improves the aerodynamics of the rotor when compared with the classical Stewart's (1976) optimization, by controlling the starting axial induction factor in the rotor plane in the process of calculating the chord and twist angle distributions. It is also observed that the model shows good efficiency when compared with the rotor designed using Stewart's (1976) model. The results, although preliminary, show good consistency and considerable importance. Other simulations are being conducted with the proposed model.

5. References

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